Capacity of a MIMO Wireless System

signal noise Gaussian var σ^2

$$\tilde{y}_i = \sigma_i \tilde{x}_i + \tilde{w}_i$$

σ_i: Gain of i^{th} channel

$E\{|\tilde{x}_i|^2\} = P_i$

P_i: Power of i^{th} transmitted symbol.

$$SNR = \frac{\sigma_i^2 P_i}{\sigma^2}$$

For i^{th} channel $\quad i = 1, 2, \ldots, t$

$$SNR_i = \frac{\sigma_i^2 P_i}{\sigma^2}$$

$$\text{Shannon rate} = \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma^2}\right)$$

$$\text{The sum rate} = \sum_{i=1}^{t} \log_2\left(1 + SNR_i\right)$$

$$\text{Sum rate} = \sum_{i=1}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma^2}\right)$$

sum of rates across t channels

$$\sum_{i=1}^{t} P_i \leq P$$

P: Transmit Power

Capacity of MIMO channel

$$= \begin{cases} \max. \sum_{i=1}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma^2}\right) \\ \text{Subject to.} \sum_{i=1}^{t} P_i = P. \end{cases}$$

Constrained Optimization problem

→ Lagrange multipliers.

$$\sum_{i=1}^{t} \log_2\left(1+\frac{\sigma_i^2 P_i}{\sigma^2}\right) + \lambda\left(P - \sum_{i=1}^{t} P_i\right)$$

Lagrange multiplier

$$\frac{\partial}{\partial P_i} = \frac{\frac{\sigma_i^2}{\sigma^2}}{1+\sigma_i \frac{P_i}{\sigma^2}} + \lambda(0-1) = 0$$

$$\Rightarrow \quad \frac{\sigma_i^2/\sigma^2}{1+\sigma_i^2 P_i/\sigma^2} = \lambda$$

$$\frac{\sigma_i^2/\sigma^2}{1+\sigma_i^2 P_i/\sigma^2} = \lambda$$

$$\Rightarrow \quad \frac{1}{\lambda} = P_i + \frac{\sigma^2}{\sigma_i^2}$$

$$\Rightarrow \quad P_i = \left(\frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2}\right)^+$$

$$P_i = \begin{cases} \frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2} & \text{if } \frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

λ — Lagrange multiplier

$$\sum_{i=1}^{t} P_i = P$$

$$\Rightarrow \sum_{i=1}^{t} \left(\frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2} \right) = P$$

Solving this equation yields λ — Lagrange multiplier

→ This optimal algorithm to achieve the capacity or maximum sum rate of the MIMO system is termed as Water Filling Algorithm